A High-Order Nodal Finite Element Formulation for Microwave Engineering

Rubén Otín

CIMNE - International Center For Numerical Methods in Engineering
Parque Tecnológico del Mediterráneo, Edificio C4 - Despacho 136S
Av. del Canal Olímpic, s/n E-08860 Castelldefels (Barcelona, Spain)
Tel: +34 93 413 70 17, Fax: +34 93 401 65 17
rotin@cimne.upc.edu

Abstract: In this work we are going to present an alternative approach to the usual finite element formulation based on edge elements and the double-curl Maxwell equations. This formulation will be called HORUS (*H*igh *O*rde*R* elements *U*n-gaged near the *S*ingularity) and it is based on high-order nodal elements and the regularized Maxwell equations. HORUS gives spurious-free solutions and well-conditioned matrices but, especial care must be taken in the neighborhood of a field singularity. At the end of the paper are given a few validation examples of HORUS applied to microwave filters.

Keywords: Finite element method, nodal elements, high-order finite elements, regularized Maxwell equations, weighted regularization and microwave engineering.

1. Introduction

The physical problem we want to solve is to find the electric field in a domain Ω, with a boundary $\partial\Omega$, produced by a divergence-free source $\mathbf{J}$ driven at a frequency of ω. The equations describing this general problem are,

$$\begin{aligned} \nabla \times \left(\frac{1}{\mu} \nabla \times \mathbf{E} \right) - \omega^2 \varepsilon \mathbf{E} = j\omega \mathbf{J} \qquad \text{in} \quad \Omega \\ \hat{\mathbf{n}} \times \mathbf{E} = 0 \qquad \text{in} \quad \Gamma \end{aligned} \tag{1}$$

where Γ is the surface of a perfect electric conductor. If we are in a open domain, the Silver-Müller radiation boundary condition must be added at infinity,

$$\lim_{r \to \infty} \oint_{\partial\Omega_r} \| \, \hat{\mathbf{n}} \times \nabla \times \mathbf{E} + (j\omega\sqrt{\varepsilon\mu})\, \mathbf{E} \, \|^2 = 0. \tag{2}$$

If $\partial\Omega$ is a waveguide port, the boundary conditions will be adapted to take into account the specific modes of the waveguide, for instance, in a rectangular waveguide port, with only the fundamental mode TE_{10} propagating holds,

$$\hat{\mathbf{n}} \times (\nabla \times \mathbf{E}) - \gamma_{10} \, (\hat{\mathbf{n}} \times \hat{\mathbf{n}} \times \mathbf{E}) = \mathbf{U} \tag{3}$$

being γ_{10} the propagation constant of the mode TE_{10} and being $\mathbf{U} = -2\gamma_{10}(\hat{\mathbf{n}} \times \hat{\mathbf{n}} \times \mathbf{E}_{TE_{10}})$ for the input port, $\mathbf{U} = 0$ for the output port [1].

This differential equation can be solved employing the *double-curl weak formulation*, that is, if we define $\mathbf{H}_0(\mathbf{curl};\Omega) := \{\, \mathbf{U} \in \mathbf{L}^2(\Omega) \;|\; \nabla \times \mathbf{U} \in \mathbf{L}^2(\Omega) \;,\; \hat{\mathbf{n}} \times \mathbf{U} = 0 \;\text{ in }\; \Gamma\}$, find a $\mathbf{E} \in \mathbf{H}_0(\mathbf{curl};\Omega)$ such that, $\forall\, \mathbf{F} \in \mathbf{H}_0(\mathbf{curl};\Omega)$ holds,

$$\int_\Omega \frac{1}{\mu} \; (\nabla \times \mathbf{E}) \cdot (\nabla \times \bar{\mathbf{F}}) \; d\Omega \;\; - \;\; \omega^2 \int_\Omega \varepsilon\, \mathbf{E} \cdot \bar{\mathbf{F}} \;\; d\Omega \;\; + \;\; \textbf{B.C.}|_{\partial\Omega} \;\; = \;\; j\omega \int_\Omega \mathbf{J} \cdot \bar{\mathbf{F}} \;\; d\Omega \tag{4}$$

where the expression $\textbf{B.C.}|_{\partial\Omega}$ is the term that takes into account the radiation boundary conditions or the modes in a waveguide port.

The typical approach, in the finite element method, is to solve the above weak formulation using *edge elements*. This edge elements, proposed by Nédélec [2], seem to be the answer to most of the drawbacks exhibited by FEM when applied to electromagnetism (see [3, 1] and references therein). With edge elements are obtained spurious-free solutions, boundary conditions are easier to implement and the normal discontinuity and tangential continuity between different media are automatically satisfied. However, using edge elements with the double-curl formulation (4) have some serious disadvantages [4, 5]. The most important flaw is the matrices produced, which are ill-conditioned and that, in problems with a high number of unknowns, can even be singular [6]. Therefore, although edge elements have several advantageous features, we can find ourselves in trouble when trying to solve problems where the use of direct methods, or good preconditioning, can be limited by our hardware. There are other valid options, like the use of *potentials* [7] or *Lagrange multipliers* [8, 9], but the number of unknowns is increased due to the presence of an extra scalar function.

Under this scenario, we want to explore new possibilities. Our objective the rest of the paper is to present a formulation that only uses the electric field as variable and that gives well-conditioned matrices.

2. Regularized Formulation with Nodal Elements

An alternative approach to solve (1) is to use an equivalent system of second-order differential equations called *regularized Maxwell equations*,

$$\begin{aligned} \nabla \times \left(\frac{1}{\mu} \nabla \times \mathbf{E}\right) - \bar{\varepsilon}\, \nabla \left(\frac{1}{\bar{\varepsilon}\varepsilon\mu} \nabla \cdot (\varepsilon \mathbf{E})\right) - \omega^2 \varepsilon \mathbf{E} = j\omega \mathbf{J} \qquad &\text{in} \quad \Omega \\ \nabla \cdot (\varepsilon \mathbf{E}) = 0 \qquad &\text{in} \quad \Gamma \\ \hat{\mathbf{n}} \times \mathbf{E} = 0 \qquad &\text{in} \quad \Gamma \end{aligned} \tag{5}$$

where Γ is again the surface of a perfect electric conductor. If we are in a open domain, the Silver-Müller radiation boundary condition (2) must be adapted to the regularization,

$$\begin{aligned} \lim_{r\to\infty} \oint_{\partial\Omega_r} \|\, \hat{\mathbf{n}} \times \nabla \times \mathbf{E} \; - \; j\omega\sqrt{\varepsilon\mu}\, (\hat{\mathbf{n}} \times \hat{\mathbf{n}} \times \mathbf{E})\, \|^2 = 0, \\ \lim_{r\to\infty} \oint_{\partial\Omega_r} |\, \nabla \cdot \mathbf{E} \; - \; j\omega\sqrt{\varepsilon\mu}\, (\hat{\mathbf{n}} \cdot \mathbf{E})\, |^2 = 0. \end{aligned} \tag{6}$$

If $\partial\Omega$ is a waveguide port, the boundary conditions also must be adapted to the regularization, for instance, in a rectangular waveguide port, with only the fundamental mode TE_{10} propagating, we must add to (3) the condition,

$$\hat{\mathbf{n}} \cdot \mathbf{E} = 0. \tag{7}$$

This differential equation can be solved employing the *regularized weak formulation*, that is, if we define $\mathbf{H}_0(\mathbf{curl}, \text{div}; \Omega) := \{ \mathbf{U} \in \mathbf{L}^2(\Omega) \ | \nabla \times \mathbf{U} \in \mathbf{L}^2(\Omega) \, , \, \nabla \cdot (\varepsilon \mathbf{U}) \in L^2(\Omega) \, , \, \hat{\mathbf{n}} \times \mathbf{U} = 0 \text{ in } \Gamma \}$, find a $\mathbf{E} \in \mathbf{H}_0(\mathbf{curl}, \text{div}; \Omega)$ such that, $\forall\, \mathbf{F} \in \mathbf{H}_0(\mathbf{curl}, \text{div}; \Omega)$ holds,

$$\begin{aligned} \int_\Omega \frac{1}{\mu} \ (\nabla \times \mathbf{E}) \cdot (\nabla \times \bar{\mathbf{F}}) \, d\Omega + \int_\Omega \frac{1}{\bar{\varepsilon}\varepsilon\mu} \ (\nabla \cdot (\varepsilon \mathbf{E})) \cdot (\nabla \cdot (\bar{\varepsilon}\, \bar{\mathbf{F}})) \, d\Omega \\ - \omega^2 \int_\Omega \varepsilon \, \mathbf{E} \cdot \bar{\mathbf{F}} \, d\Omega \ + \ \textbf{R.B.C.}|_{\partial\Omega} = \ j\omega \int_\Omega \mathbf{J} \cdot \bar{\mathbf{F}} \, d\Omega \end{aligned} \tag{8}$$

where the expression $\textbf{R.B.C.}|_{\partial\Omega}$ is the term, properly adapted to the regularization, that takes into account the radiation boundary conditions or the modes in a waveguide port.

In [10] is demonstrated the equivalence between the *classical* problem (1), (4) and the *regularized* problem (5),(8). The regularized formulation has the characteristics required for an efficient FEM simulation, that is, it calculates spurious-free solutions, with $\mathbf{E}$ as the only unknown, and it produces well-conditioned matrices. Nodal elements can be used as the finite element base but, being careful in considering explicitly the discontinuities between different media, as is done in [11]. Also, one has to be careful in defining the normals when two or more dialectics intersect.

3. Field Singularities and the Regularized Formulation

In [10] is set that solving (8) is completely equivalent to solve the Maxwell equations, almost in a continuous level. But, if (8) is discretized with nodal finite elements in a non-convex polyhedral domain, the situation changes. The problem is that the vectorial space spanned by the nodal basis is included in $\mathbf{H}_0^1(\Omega) := \{ \mathbf{U} \in \mathbf{H}^1(\Omega) \mid \hat{\mathbf{n}} \times \mathbf{U} = 0 \text{ in } \Gamma \}$. On the other hand, $\mathbf{H}_0^1(\Omega)$ is strictly included in $\mathbf{H}_0(\mathbf{curl}, \text{div}; \Omega)$ for non-convex polyhedral domains and, moreover, $\mathbf{H}_0^1(\Omega)$ is closed in $\mathbf{H}_0(\mathbf{curl}, \text{div}; \Omega)$ [12, 13]. Then, sometimes, we can have a solution of (8) that belongs to $\mathbf{H}_0(\mathbf{curl}, \text{div}; \Omega)$ but not to $\mathbf{H}_0^1(\Omega)$, as happens, for instance, when the electric field is singular near the corners or edges of a perfect electric conductor. In this case, it will be impossible to approximate the field using nodal elements, in fact, it will be impossible using any $\mathbf{H}^1$-conforming finite element discretization [14]. Summarizing, if the electric field is singular in some point of the domain, the solution obtained with nodal elements and (8) will be *globally wrong* no matter the element size or the polynomial order used in the discretization.

4. Weighted Regularization and HORUS

The *weighted regularized Maxwell equations method* (WRME) [13] is a robust formulation that overcomes the problem stated above. This is achieved by multiplying the divergence term of (8) by a geometry dependant weight. This weight tends to zero when approaching to a field singularity. WRME gives well-conditioned matrices and nodal elements can be used as the finite element base.

HORUS (*H*igh *O*rde*R* elements *U*n-gaged near the *S*ingularity) is a simplification of WRME. In HORUS there is no need to calculate any geometry dependant weight. To achieve results similar to those in WRME, the divergence term of (8) is removed from the elements that are in contact with the edges and corners of dielectrics and perfect electric conductors, and also, from the elements that are in contact with the intersection of several dielectrics. In [15, 16] the same strategy is followed for quasi-static problems.

5. Validation Examples

To check the accuracy of HORUS was calculated the scattering parameters of some microwave devices. Fig.1, Fig.2 and Fig.3 are some examples of the simulations performed with HORUS. All the results were obtained using a bi-conjugate gradient solver with a diagonal preconditioner. The number of iterations needed to reach a relative error less than 10^{-4} was always less than 1% of the total number of unknowns.

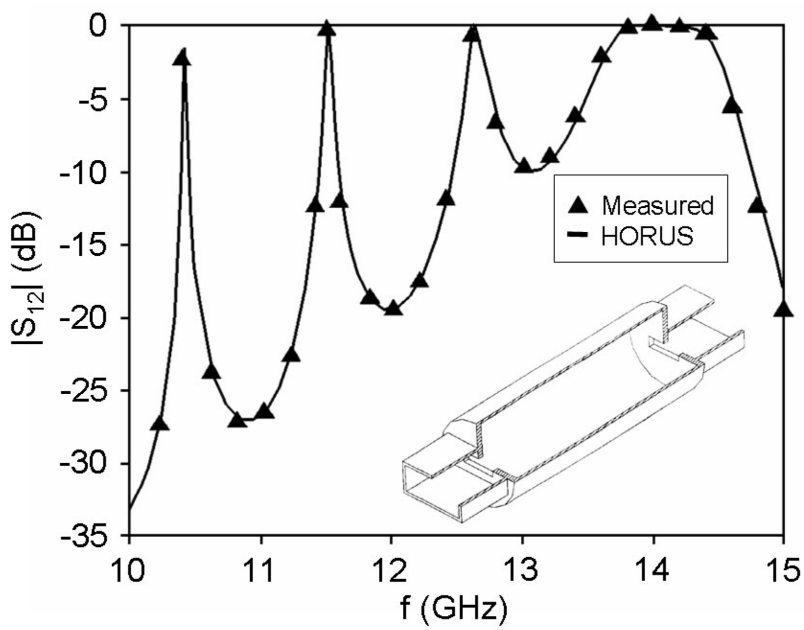

Fig. 1. Circular cavity filter measured in [17]. HORUS used 3rd order nodal elements. The divergence term of (8) was removed from the elements in contact with the edges of the coupling slots.

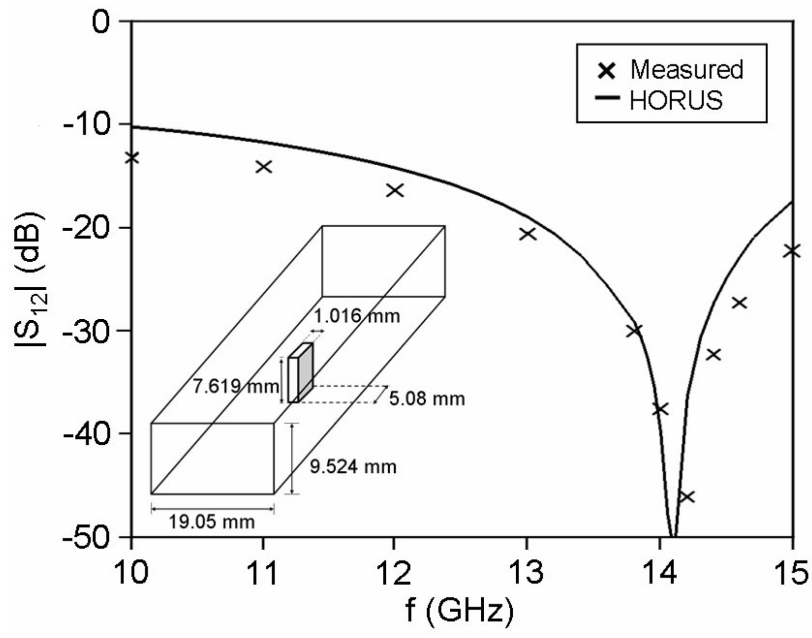

Fig. 2. Ridge waveguide measured in [18]. HORUS used 3rd order nodal elements. The divergence term of (8) was removed from the elements in contact with the edges of the ridge.

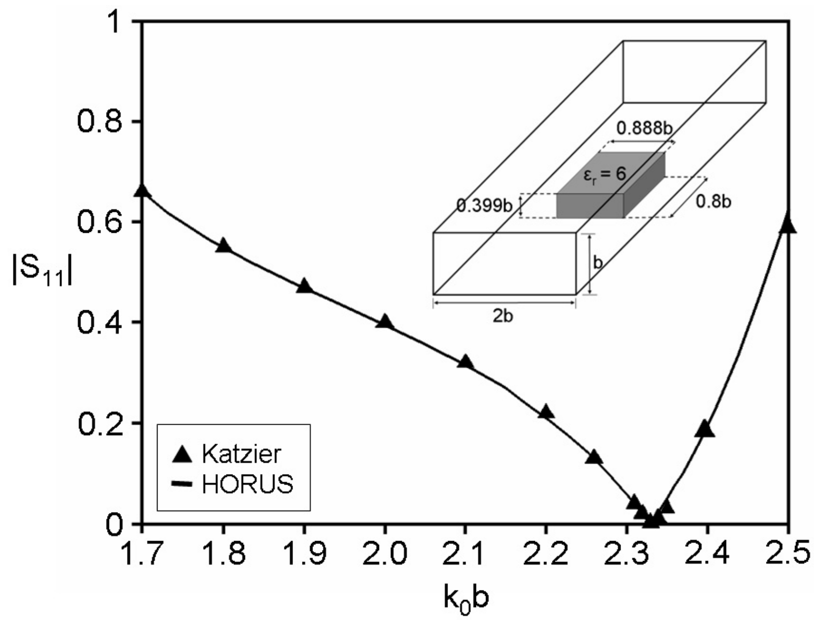

Fig. 3. Method in [19]. HORUS used 3rd order nodal elements. The divergence term of (8) was removed from the elements in contact with the edges of the dielectric.

6. Conclusion

In this paper has been presented HORUS, a nodal finite element formulation that solves the regularized Maxwell equations. To overcome the problem raised by the presence of field singularities, the divergence term of (8) is removed from the elements in contact with the edges and corners of dielectrics and perfect electric conductors, and also, from the elements in contact with the intersection of several dielectrics. Currently more simulations are being performed, with different geometries and boundary conditions, to check the robustness of the method.

Acknowledgments

This work has been partially founded by the Spanish Ministry of Industry, Tourism and Commerce in the frame of the project PROFIT SANTTRA (*Sistema de ANTenas para Transceptores de RAdio*, Ref: FIT-330210-2006-44) based on the European project PIDEA-EUREKA SMART (*SMart Antennas system for Radio Transceivers*).

References

[1] J. Jin, *The Finite Element Method in Electromagnetics*. John Wiley & Sons, 2002.
[2] J. C. Nedelec, "Mixed finite elements in R^3," *Numerische Mathematik*, vol. 35, pp. 315–341, 1980.
[3] M. Salazar-Palma, T. K. Sarkar, L.-E. G. Castillo, T. Roy, and A. Djordjevic, *Iterative and Self-Adaptive Finite-Elements in Electromagnetic Modeling*. Artech House Publishers, 1998.
[4] G. Mur, "The fallacy of edge elements," *IEEE Transactions on Magnetics*, vol. 34, pp. 3244–3247, 1998.
[5] ——, "Edge elements, their advantages and their disadvantages," *IEEE Transactions on Magnetics*, vol. 30, pp. 3552–3557, 1994.

[6] I. E. Lager, “Finite element modelling of static and stationary electric and magnetic fields,” Ph.D. dissertation, Delft Univesity, 1996.

[7] W. E. Boyse, D. R. Lynch, K. D. Paulsen, and G. N. Minerbo, “Nodal-based finite-element modeling of Maxwell’s equations,” *IEEE Transactions on Antennas and Propagation*, vol. 40, pp. 642–651, 1992.

[8] P. C. Jr., “Augmented formulations for solving Maxwell equations,” *Computer Methods in Applied Mechanics and Engineering*, vol. 194, pp. 559–586, 2005.

[9] R. Hiptmair, “Finite elements in computational electromagnetism,” *Acta Numerica*, pp. 237–339, 2002.

[10] C. Hazard and M. Lenoir, “On the solution of the time-harmonic scattering problems for Maxwell’s equations,” *SIAM Journal on Mathematical Analysis*, vol. 27, pp. 1597–1630, 1996.

[11] K. D. Paulsen, D. R. Lynch, and J. W. Strohbehn, “Three-dimensional finite, boundary, and hybrid element solutions of the Maxwell equations for lossy dielectric media,” *IEEE Transactions on Microwave Theory and Techniques*, vol. 36, pp. 682–693, 1988.

[12] M. Costabel, “A coercive bilinear form for Maxwell’s equations,” *Journal of Mathematical Analysis and Applications*, vol. 157, no. 2, pp. 527–541, 1991.

[13] M. Costabel and M. Dauge, “Weighted regularization of Maxwell equations in polyhedral domains,” *Numerische Mathematik*, vol. 93, no. 2, pp. 239–277, 2002.

[14] ——, “Maxwell and Lamé eingevalues on polyhedra,” *Mathematical Methods in Applied Science*, vol. 22, pp. 243–258, 1999.

[15] K. Preis, O. Bíró, and I. Ticar, “Gauged current vector potential and reetrant corners in the FEM analysis of 3D eddy currents,” *IEEE Transactions on Magnetics*, vol. 36, pp. 840–843, 2000.

[16] M. Kaltenbacher and S. Reitzinger, “Appropiate finite-element formulation for 3-D electromagnetic-field problems,” *IEEE Transactions on Magnetics*, vol. 38, pp. 513–516, 2002.

[17] J. R. Montejo-Garai and J. Zapata, “Full-wave design and realization of multicoupled dual-mode circular waveguide filters,” *IEEE Transactions on Microwave Theory and Techniques*, vol. 43, no. 6, pp. 1290–1297, 1995.

[18] R. R. Mansour, R. S. K. Tong, and R. H. Machpie, “Simplified description of the field distribution in finlines and ridge waveguides and its application to the analysis of E-plane discontinuities,” *IEEE Transactions on Microwave Theory and Techniques*, vol. 36, no. 12, pp. 1825–1832, 1988.

[19] H. Katzier, “Streuverhalten elektromagnetischer Wellen bei sprunghaften Übergängen geschirmter dielektrischer leitungen,” *Arch. Elek. Übertragung*, vol. 38, pp. 290–296, 1984.