Regularized Maxwell Equations and Nodal Finite Elements for Electromagnetic Field Computations

Ruben Otin

CIMNE - International Center For Numerical Methods in Engineering
Parque Tecnológico del Mediterráneo, Edificio C3 - Despacho 206
Av. del Canal Olímpic, s/n E-08860 Castelldefels (Barcelona, Spain)
Tel: +34 93 413 70 17, Fax: +34 93 401 65 17
rotin@cimne.upc.edu

This paper presents an alternative approach to the usual finite element formulation based on edge elements and the double-curl Maxwell equations. This alternative approach is based on nodal elements and the regularized Maxwell equations. The advantage is that, without adding extra unknowns, like Lagrange multipliers, it provides spurious-free solutions and well-conditioned matrices. The drawback is that a globally wrong solution is obtained when the electromagnetic field has a singularity in the problem domain. The main objective of this work is to obtain accurate solutions with nodal elements and the regularized formulation even in the presence of electromagnetic field singularities.

Keywords: Finite element method, regularized Maxwell equations, weighted regularization, singularities, intersection of dielectrics and nodal elements.

Introduction

The typical approach when solving a general electromagnetic problem with the finite element method (FEM) is to use edge elements and the double-curl Maxwell equations. These edge elements, proposed by Nédélec [1], seem to be the answer to most of the drawbacks exhibited by FEM when applied to electromagnetism (see [2, 3] and references therein). With edge elements, spurious-free solutions are obtained, boundary conditions are easier to implement and the normal discontinuity and tangential continuity between different media are automatically satisfied. In addition, they present a better behavior in non-convex domains than Lagrangian elements [4]. However, using edge elements with the double-curl formulation has also disadvantages [5, 6]. The most important flaw is the matrices produced, which are ill-conditioned and, in problems with a high number of unknowns, can even be singular [7]. The use of potentials or Lagrange multipliers can improve the conditioning of the matrix [8, 9] but the number of unknowns is increased by the presence of these scalar functions. Therefore, although edge elements present several advantageous features, we can have troubles when trying to solve big problems where the use of direct methods, or good preconditioning, can be limited by our computer hardware. Because of this, it would be desirable to explore alternative FEM formulations which provide us with matrices easy to solve by means of iterative solvers.

The alternative approach proposed in this work is based on nodal elements and the regularized Maxwell equations [10, 11]. The good point of this proposal is that it provides spurious-free solutions with well-conditioned matrices and, moreover, only the three components of $\mathbf{E}$, or $\mathbf{H}$, are the unknowns, that is, there is no need of extra functions, like Lagrange multipliers or scalar potentials. On the other hand, new difficulties arise that were not present in the classical formulation. The main drawback is that, if the electromagnetic field has a singularity in the problem domain, a *globally wrong* solution is obtained. Also, boundary conditions and field discontinuities are more laborious to implement. In this paper it is explained how to overcome these difficulties. The main objective is to demonstrate that accurate solutions can be obtained using nodal elements and the regularized formulation. The comparative performance of the alternative formulations, *nodal-regularized* versus the classical formulation *edge-double-curl* will be the topic of future work.

In the first two sections, the classical double-curl formulation and the regularized Maxwell formulation are exposed, both adapted to electromagnetic problems in frequency domain. The formulations are written to emphasize their differences in the differential and weak form and also in the boundary conditions. This method can use $\mathbf{E}$ or $\mathbf{H}$ as the primary unknown but, in this work, for simplicity, we use the $\mathbf{E}$ field only.

In the next section, it is explained how to deal with field discontinuities and nodal elements. It is not only explained how to solve problems with a discontinuity surface present, but also how to work with the intersection of three or more different materials.

Once we are acquainted with the regularized formulation, the problem of the singularities is described: why it is produced and how to overcome this critical question.

Finally, in the last sections, several examples in 2D and 3D are shown to check the accuracy of the method.

Double-Curl Maxwell Equations and Edge Elements

The generic problem we want to solve is to find the electric field $\mathbf{E}$ in a domain Ω, with boundary $\partial\Omega$, produced by a divergence-free source $\mathbf{J}$ driven at a frequency ω. The equations describing this situation, the so-called *double-curl Maxwell equations*, are

$$\begin{aligned} \nabla \times \left(\frac{1}{\mu} \nabla \times \mathbf{E} \right) - \omega^2 \varepsilon \mathbf{E} = -j\omega \mathbf{J} \qquad & \text{in} \quad \Omega \\ \hat{\mathbf{n}} \times \mathbf{E} = 0 \qquad & \text{in} \quad \Gamma \end{aligned} \tag{1}$$

where Γ is the surface of a perfect electric conductor. If we are in an open domain, the Silver-Müller radiation boundary condition must be added at infinity:

$$\lim_{r\to\infty} \oint_{\partial\Omega_r} \| \, \hat{\mathbf{n}} \times \nabla \times \mathbf{E} \, - \, (j\omega\sqrt{\varepsilon\mu}) \, \mathbf{E} \, \|^2 = 0 \tag{2}$$

If $\partial\Omega$ is a waveguide port, the boundary conditions will be adapted to take into account the specific modes of the waveguide. For instance, in a rectangular waveguide port, with only the fundamental mode TE_{10} propagating,

$$\hat{\mathbf{n}} \times (\nabla \times \mathbf{E}) \, - \, \gamma_{10} \, (\hat{\mathbf{n}} \times \hat{\mathbf{n}} \times \mathbf{E}) = \mathbf{U} \tag{3}$$

holds, where γ_{10} is the propagation constant of the fundamental mode $\mathbf{E}_{10}$ and

$$\begin{aligned} &\mathbf{U} = -2\gamma_{10}(\hat{\mathbf{n}} \times \hat{\mathbf{n}} \times \mathbf{E}_{10}), \\ &\mathbf{U} = 0 \end{aligned} \tag{4}$$

for the input and the output port, respectively [3].

The above set of equations can be solved using an equivalent weak formulation, that is, if we define

$$\mathbf{H}_0(\mathbf{curl};\Omega) := \{ \, \mathbf{F} \in \mathbf{L}^2(\Omega) \ | \ \nabla \times \mathbf{F} \in \mathbf{L}^2(\Omega), \ \ \hat{\mathbf{n}} \times \mathbf{F} = 0 \ \ \text{in} \ \ \Gamma \} \tag{5}$$

solving equation (1) is equivalent to finding an $\mathbf{E} \in \mathbf{H}_0(\mathbf{curl};\Omega)$ such that, $\forall \, \mathbf{F} \in \mathbf{H}_0(\mathbf{curl};\Omega)$ holds

$$\int_\Omega \frac{1}{\mu} (\nabla \times \mathbf{E}) \cdot (\nabla \times \bar{\mathbf{F}}) \, - \, \omega^2 \int_\Omega \varepsilon \, \mathbf{E} \cdot \bar{\mathbf{F}} \, + \, \mathbf{B.C.}|_{\partial\Omega} = -j\omega \int_\Omega \mathbf{J} \cdot \bar{\mathbf{F}} \tag{6}$$

where the expression $\mathbf{B.C.}|_{\partial\Omega}$ is the term that takes into account the radiation boundary conditions or the modes in a waveguide port. This weak formulation, discretized with edge elements, is the classical way to use the finite element method when applied to electromagnetic field problems. The advantages and disadvantages of this approach were cited in the introduction of this paper.

Regularized Maxwell Equations and Nodal Elements

An alternative to solve (1) is to use an equivalent system of second-order differential equations, the so-called *regularized Maxwell equations*:

$$\begin{aligned} \nabla\times\left(\frac{1}{\mu}\nabla\times\mathbf{E}\right)-\bar{\varepsilon}\,\nabla\left(\frac{1}{\bar{\varepsilon}\varepsilon\mu}\nabla\cdot(\varepsilon\mathbf{E})\right)-\omega^2\varepsilon\mathbf{E}=-j\omega\mathbf{J} &\quad \text{in}\quad \Omega\\ \nabla\cdot(\varepsilon\mathbf{E})=0 &\quad \text{in}\quad \Gamma\\ \hat{\mathbf{n}}\times\mathbf{E}=0 &\quad \text{in}\quad \Gamma \end{aligned} \tag{7}$$

where Γ is again the surface of a perfect electric conductor and $\bar{\varepsilon}$ is the complex conjugate of ε. If we are in an open domain, the Silver-Müller radiation boundary condition (2) must be adapted to the regularization

$$\begin{aligned} \lim_{r\to\infty}\oint_{\partial\Omega_r}\|\,\hat{\mathbf{n}}\times\nabla\times\mathbf{E}\,+\,j\omega\sqrt{\varepsilon\mu}\,(\hat{\mathbf{n}}\times\hat{\mathbf{n}}\times\mathbf{E})\,\|^2=0\\ \lim_{r\to\infty}\oint_{\partial\Omega_r}|\,\nabla\cdot\mathbf{E}\,+\,j\omega\sqrt{\varepsilon\mu}\,(\hat{\mathbf{n}}\cdot\mathbf{E})\,|^2=0 \end{aligned} \tag{8}$$

If $\partial\Omega$ is a waveguide port, the boundary conditions must also be adapted to the regularization, for instance, in a rectangular waveguide port, with only the fundamental mode TE_{10} propagating, we must add to (3) the condition

$$\hat{\mathbf{n}}\cdot\mathbf{E}=0. \tag{9}$$

The above differential equation can be solved using an equivalent weak formulation, that is, if we define

$$\begin{aligned} \mathbf{H}_0(\mathbf{curl},\text{div};\Omega):=\{\,\mathbf{F}\in\mathbf{L}^2(\Omega)\,|\,\nabla\times\mathbf{F}\in\mathbf{L}^2(\Omega),\,\nabla\cdot(\varepsilon\mathbf{F})\in L^2(\Omega),\\ \hat{\mathbf{n}}\times\mathbf{F}=0\ \text{ in }\ \Gamma\,\}, \end{aligned} \tag{10}$$

solving equation (7) is equivalent to finding an $\mathbf{E}\in\mathbf{H}_0(\mathbf{curl},\text{div};\Omega)$ such that, $\forall\,\mathbf{F}\in\mathbf{H}_0(\mathbf{curl},\text{div};\Omega)$

$$\begin{aligned} \int_\Omega\frac{1}{\mu}\ (\nabla\times\mathbf{E})\cdot(\nabla\times\bar{\mathbf{F}})\ +\ \int_\Omega\frac{1}{\bar{\varepsilon}\varepsilon\mu}\ (\nabla\cdot(\varepsilon\mathbf{E}))\cdot(\nabla\cdot(\bar{\varepsilon}\,\bar{\mathbf{F}}))\\ -\ \omega^2\int_\Omega\varepsilon\,\mathbf{E}\cdot\bar{\mathbf{F}}\ +\ \textbf{R.B.C.}|_{\partial\Omega}\ =\ -j\omega\int_\Omega\mathbf{J}\cdot\bar{\mathbf{F}} \end{aligned} \tag{11}$$

holds. The expression $\textbf{R.B.C.}|_{\partial\Omega}$ is the term, properly adapted to the regularization, that takes into account the radiation boundary conditions or the modes in a waveguide port.

In [10], the equivalence between the *classical* problem (1), (6) and the *regularized* problem (7),(11) is demonstrated. The regularized formulation has the characteristics required for an efficient FEM simulation, that is, it calculates spurious-free solutions, with **E** as the only unknown, and it produces well-conditioned matrices. Nodal elements can be used as the finite element base but, being careful in considering explicitly the discontinuities between different media, as explained in the next section.

Although equation (11) looks like a penalized method, the regularized formulation has no undetermined constants and with the help of the extra boundary conditions the problem is well-posed. It is worth to emphasize the importance of the extension of the boundary conditions in the regularized formulation: if these extra boundary conditions are omitted we can increase severely the number of iterations needed for an iterative solver to achieve convergence or even obtain spurious solutions.

Nodal Elements and Field Discontinuities

As mentioned in the previous section, nodal elements are used along with the regularized formulation. Due to the fact that these elements impose normal and tangential continuity, we must consider explicitly the discontinuities between different media. To do so, we employ the technique explained in [12]. This technique consists in defining two different nodes, one on each side of the discontinuity and, during the assembly procedure, relate the two nodes as follows,

$$\begin{pmatrix} E_x^+ \\ E_y^+ \\ E_z^+ \end{pmatrix} = \begin{pmatrix} n_x^2\xi + 1 & n_x^2 n_y^2 \xi & n_x^2 n_z^2 \xi \\ n_x^2 n_y^2 \xi & n_y^2 \xi + 1 & n_y^2 n_z^2 \xi \\ n_x^2 n_z^2 \xi & n_y^2 n_z^2 \xi & n_z^2 \xi + 1 \end{pmatrix} \begin{pmatrix} E_x^- \\ E_y^- \\ E_z^- \end{pmatrix} \tag{12}$$

being

$$\xi = \frac{\epsilon^- - j(\sigma^-/\omega)}{\epsilon^+ - j(\sigma^+/\omega)} - 1,$$

$\mathbf{n} = (n_x, n_y, n_z)$ is the unit normal at the surface and superscripts ”+” and ”-” denote each side of the discontinuity surface. In this procedure, the node in side ”+” is removed from the total linear system with the help of (12) and only the unknowns in the side ”-” are solved for. No extra unknowns are added and the symmetry of the total FEM matrix is retained if the side ”+” is removed using (12) and its transpose [13].

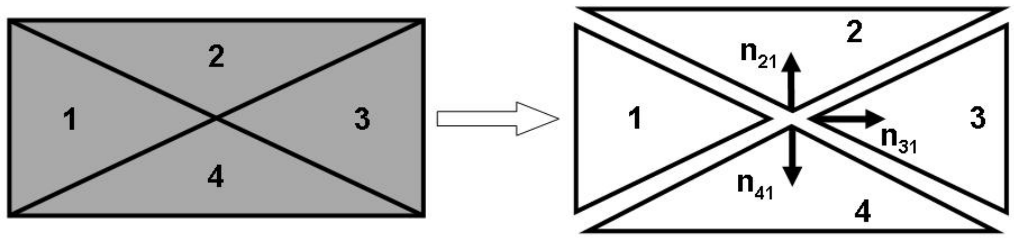

Figure 1: Intersection of four different materials. The interface between different materials is considered as separated surfaces related by (12). The vectors $\mathbf{n}_{21}$, $\mathbf{n}_{31}$ and $\mathbf{n}_{41}$ are the unit normals to the surfaces limiting the volume of materials 2, 3 and 4, respectively. In the intersection, we apply (12) to the pairs 2-1, 3-1 and 4-1.

Although this technique works well for discontinuity surfaces, we need a procedure to deal with the intersection of three or more different materials. An attempt was made in [14], but the finite element bases appearing there do not belong to $\mathbf{H}_0(\mathbf{curl}, \text{div}; \Omega)$ and can not be used to discretize (11). Our approach to overcome this setback consists in a simple extension of the method presented at the beginning of this section. For instance, in a situation like the one shown in figure 1, we define four different nodes at the center. One of the nodes defined at the center play the role of ”-” in (12). In this case the selected node is in material 1. Then, during the assembly procedure, we apply (12) to the pairs 2-1, 3-1 and 4-1, with unit normals $\mathbf{n}_{21}$, $\mathbf{n}_{31}$ and $\mathbf{n}_{41}$ respectively. Nodes 2,3 and 4 are removed from the total linear system and only the unknowns of node 1 are solved. The best choice for the role of ”-” is the node that is in the material with the smallest $|\epsilon - j(\sigma/\omega)|$. It was observed that this option gives always the lowest number of iterations when solving the total FEM matrix with iterative solvers.

Regularized Formulation and Field Singularities

In [10], it is shown that solving (11) analytically is equivalent to solving (6). However, we must be careful when solving (11) numerically with nodal finite elements in a non-convex polyhedral domain. If $\mathbf{V}_h$ is the vectorial space spanned by K nodal basis functions $N_i(\mathbf{r})$,

$$\mathbf{V}_h := \{\, \mathbf{u}_h \,|\, \mathbf{u}_h = \hat{\mathbf{x}} \sum_{i=1}^{K} c_{x_i} N_i(\mathbf{r}) + \hat{\mathbf{y}} \sum_{i=1}^{K} c_{y_i} N_i(\mathbf{r}) + \hat{\mathbf{z}} \sum_{i=1}^{K} c_{z_i} N_i(\mathbf{r}) \,,\; c_i \in \mathbb{C} \,\}$$

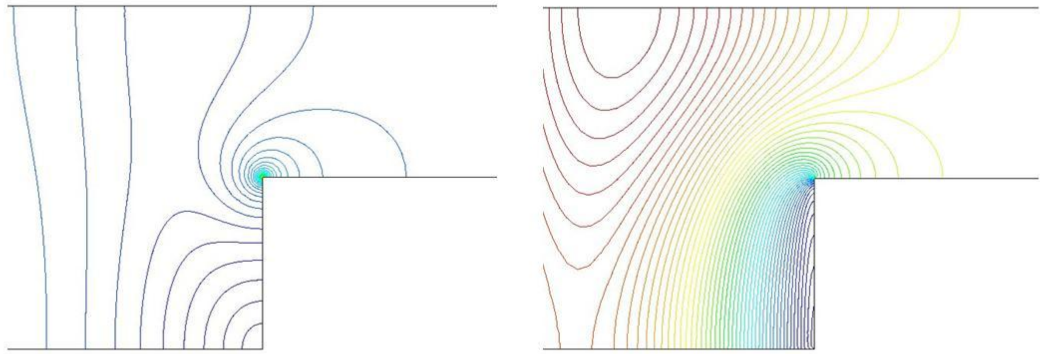

Figure 2: $\|\mathbf{E}\|$ in a waveguide step discontinuity. Left: reference solution. Right: same problem solved with the regularized formulation.

and

$$\mathbf{H}_0^1(\Omega) := \{\, \mathbf{F} \in L_2(\Omega) \,|\, \frac{\partial \mathbf{F}}{\partial x}, \frac{\partial \mathbf{F}}{\partial y}, \frac{\partial \mathbf{F}}{\partial z} \in L_2(\Omega)\,,\ \hat{\mathbf{n}} \times \mathbf{F} = 0 \text{ in } \Gamma \,\},$$

is clear that $\mathbf{V}_h$ is included in $\mathbf{H}_0^1(\Omega)$. On the other hand, for non-convex polyhedral domains, $\mathbf{H}_0^1(\Omega)$ is strictly included in $\mathbf{H}_0(\mathbf{curl}, \text{div}; \Omega)$ and, moreover, $\mathbf{H}_0^1(\Omega)$ is closed in $\mathbf{H}_0(\mathbf{curl}, \text{div}; \Omega)$ [15, 11]. As a consequence of this theorem, if **E**, the analytical solution of (11), belongs to $\mathbf{H}_0(\mathbf{curl}, \text{div}; \Omega)$ but not to $\mathbf{H}_0^1(\Omega)$, then, it is impossible to approximate **E** using nodal elements. In fact, such an approximation is impossible using any $\mathbf{H}^1$-conforming finite element discretization [16]. This situation happens, for instance, when the electric field is singular in the corners or edges of a perfect electric conductor. In other words, for non-convex polyhedral domains we have

$$\mathbf{V}_h \subseteq \mathbf{H}_0^1(\Omega) \subset \mathbf{H}_0(\mathbf{curl}, \text{div}; \Omega).$$

If the electric field is singular at some point of the domain, the analytical solution **E** belongs to $\mathbf{H}_0(\mathbf{curl}, \text{div}; \Omega)$ but not to $\mathbf{H}_0^1(\Omega)$. This means that **E** can not be approximated with nodal elements and (11), because any approximation of **E**, no matter the element size or the polynomial order used in the discretization, will belong to $\mathbf{H}_0^1(\Omega)$. In fact, what is being approximated using (11) and nodal elements is the Garlekin projection of **E** onto $\mathbf{H}_0^1(\Omega)$, which is, in general, globally different to **E**. An example of this behavior is shown in figure 2.

A good option to overcome the problem with the singularities in the regularized formulation is to follow the *weighted regularized Maxwell equations method* (WRME) explained in [11]. There are other possibilities, like [17, 18] or [19], but the WRME is more general and robust. In the WRME we have to multiply the divergence term of (11) by a geometry dependant weight. This weight tends to zero when approaching to a field singularity. To be more specific, the weight τ is

$$\tau := \left(\prod_{corners} r^{\gamma_c} \right) \cdot \left(\prod_{edges} \rho^{\gamma_e} \right), \tag{13}$$

where r and ρ are the distances from a point in the domain to the corner or edge where the field is singular. The coefficients γ_c and γ_e only depend on the geometry and can be calculated theoretically [11]. The WRME defines the space

$$\mathbf{X} := \{ \, \mathbf{u} \in \mathbf{H}_0(\mathbf{curl};\Omega) \mid \tau(\nabla \cdot \mathbf{u}) \in L^2_{loc}(\Omega) \, \},$$

and solves the problem of finding an $\mathbf{E} \in \mathbf{X}$ such that, $\forall \, \mathbf{F} \in \mathbf{X}$ holds

$$\begin{aligned} \int_\Omega \frac{1}{\mu} \; (\nabla \times \mathbf{E}) \cdot (\nabla \times \bar{\mathbf{F}}) \;\; + \;\; \int_\Omega \frac{\tau}{\bar{\varepsilon}\varepsilon\mu} \; (\nabla \cdot (\varepsilon \mathbf{E})) \cdot (\nabla \cdot (\bar{\varepsilon} \, \bar{\mathbf{F}})) \\ - \;\omega^2 \int_\Omega \varepsilon \, \mathbf{E} \cdot \bar{\mathbf{F}} \;\; + \;\; \textbf{W.R.B.C.}|_{\partial\Omega} \;\; = \;\; -j\omega \int_\Omega \mathbf{J} \cdot \bar{\mathbf{F}}, \end{aligned} \tag{14}$$

where the expression **W.R.B.C.**$|_{\partial\Omega}$ is the term, properly adapted to the weighted regularization, that takes into account the radiation boundary conditions or the modes in a waveguide port. In [11], it is demonstrated that solving (14) is equivalent to solving the Maxwell equations and also that $\mathbf{H}^1_0(\Omega)$ is dense in $\mathbf{X}$. These results imply that nodal elements converge to the right electromagnetic field solution even in the presence of singularities.

The method proposed in this work is a simplification of WRME. Instead of (13), the divergence term of (11) is multiplied by a weight which is equal to zero in the elements near a singularity and equal to one in the rest. The same idea appears in [20] for eddy current problems with the $\mathbf{T} - \Omega$ formulation and also in [21] for magnetostatic problems with the potential **A** formulation. It is worth to mention that WRME always needs finer meshes, and more iterations of the iterative solvers, to achieve the same accuracy than the simplified formulation. This was observed for the 2D problems in the next section. No testing of the WRME for 3D problems was performed.

Henceforth, when a problem is said to be solved with 1 UL (*U*ngaged *L*ayer), it means that only the elements with a node in contact with a singularity have the weight equal to zero. If a problem is said to be solved with 2 UL, it means that the weight is set to zero in the elements with a node in contact with a singularity and also, in the elements that have a node in contact with the elements of the first layer. An equivalent definition is applied for 3 UL, 4 UL and so on. Only the elements in the UL have the weight equal to zero, for the rest the weight is equal to one. These UL must be applied in all the places where the field is singular. If we keep a singularity point without this treatment, the simulation can give a globally wrong solution. To know the places where the field is singular we only need to analyze the geometry and look for [22]:

- Reentrant corners and edges of perfect electric conductors.
- Corners and edges of dielectrics.
- Intersection of several dielectrics.

Once these points are located, the number of UL depends on the size and order of the elements, as shown in the next sections. If, for some reason, very small elements are required around the singularity, the number of UL must be increased. Not doing so is equivalent to use (11) instead of (14). To distinguish the simplified formulation from the WRME, we will call the method HORUS (*H*igh *O*rde*R* elements *U*n-gaged near the *S*ingularity).

Validation Examples in 2D

This section shows some 2D configurations used to check the accuracy of HORUS (figures 3, 4, 5 and 6). The geometries employed are four different discontinuities in a parallel-plate waveguide. The problems are driven by a current density $J_y = 1$ A/m^2 at a frequency of 10 GHz. The walls of the waveguide are perfect electric conductors (PEC). The discontinuous lines at the sides symbolize the application of the 1st order absorbing boundary conditions (1st ABC). The reference solution **E** = (E_x,E_y) is obtained by the differentiation of H_z with the method explained in [3](chapter 4). On the other hand, HORUS calculates **E** = (E_x,E_y) directly from (14), with the weight τ as it is explained in the previous section. The unit normals at the nodes placed in singular corners are calculated as a geometric average and they are used in (12), or in the PEC condition, like in any other node [13]. HORUS was tested with triangular nodal elements from 1st to 6th order. From 3rd to 6th order elements and 1 UL accurate solutions were obtained even near the singularity. With 2nd order elements and 3 UL correct solutions were obtained, too. With 1st order elements it was not possible to obtain accurate solutions even with 6 UL. In the graphs of figures 3, 4, 5 and 6, it can be seen that the reference solution and HORUS overlap perfectly even in the neighborhood of the singularity. In the same graphs, the result of using (11) without taking into account the effect of the singularities is also represented (graph *Regularized ME*).

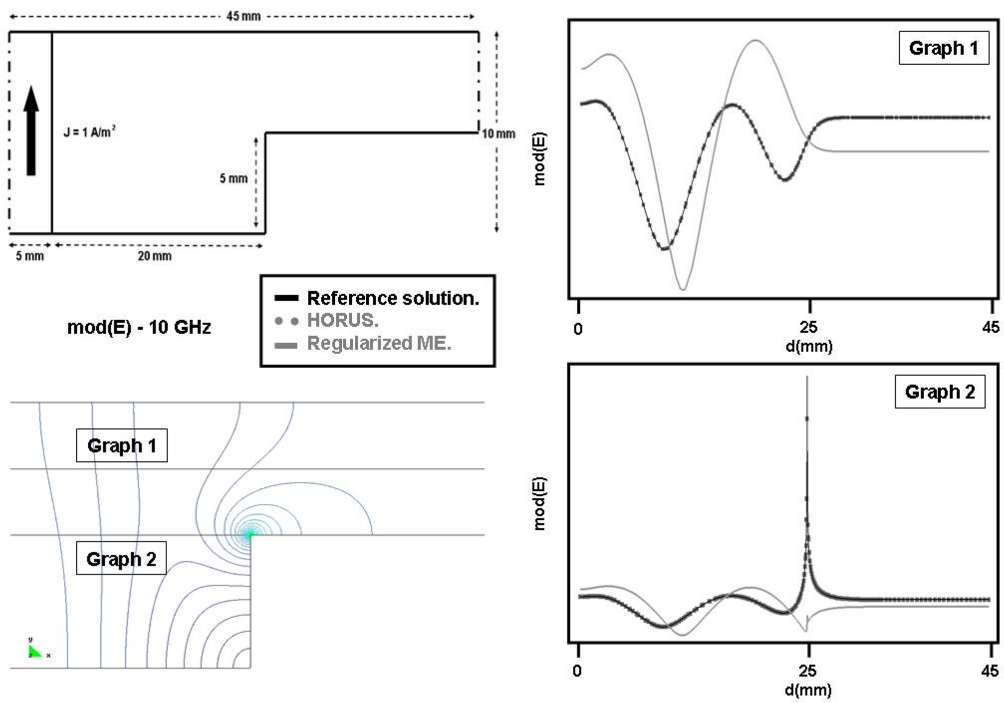

Figure 3: Step discontinuity in a parallel-plate waveguide.

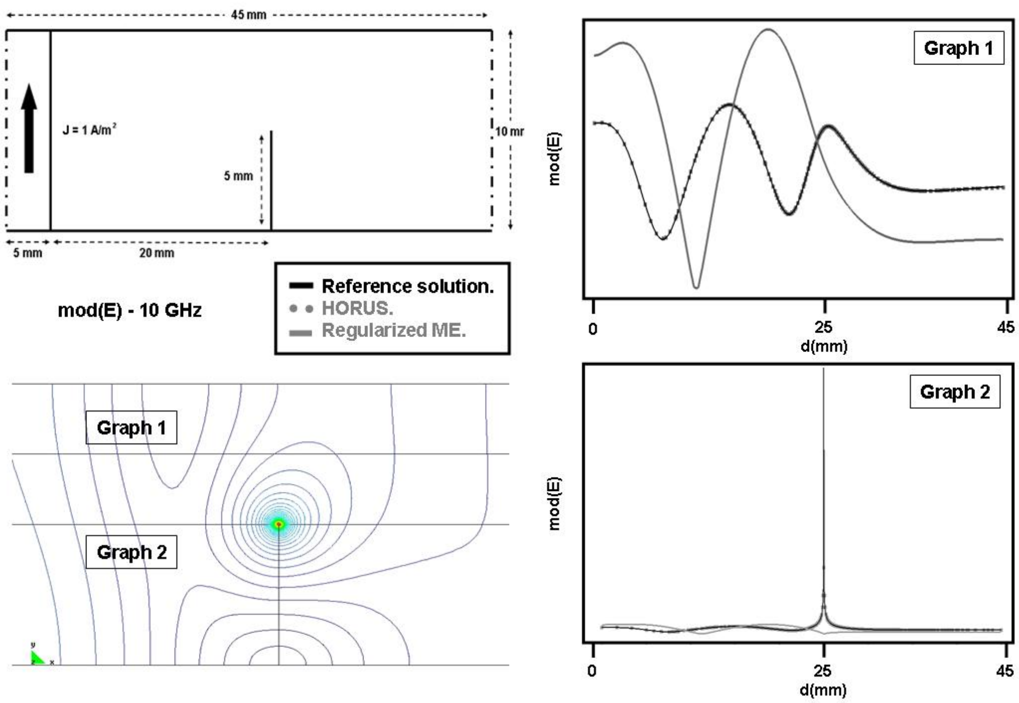

Figure 4: Sheet singularity in a parallel-plate waveguide.

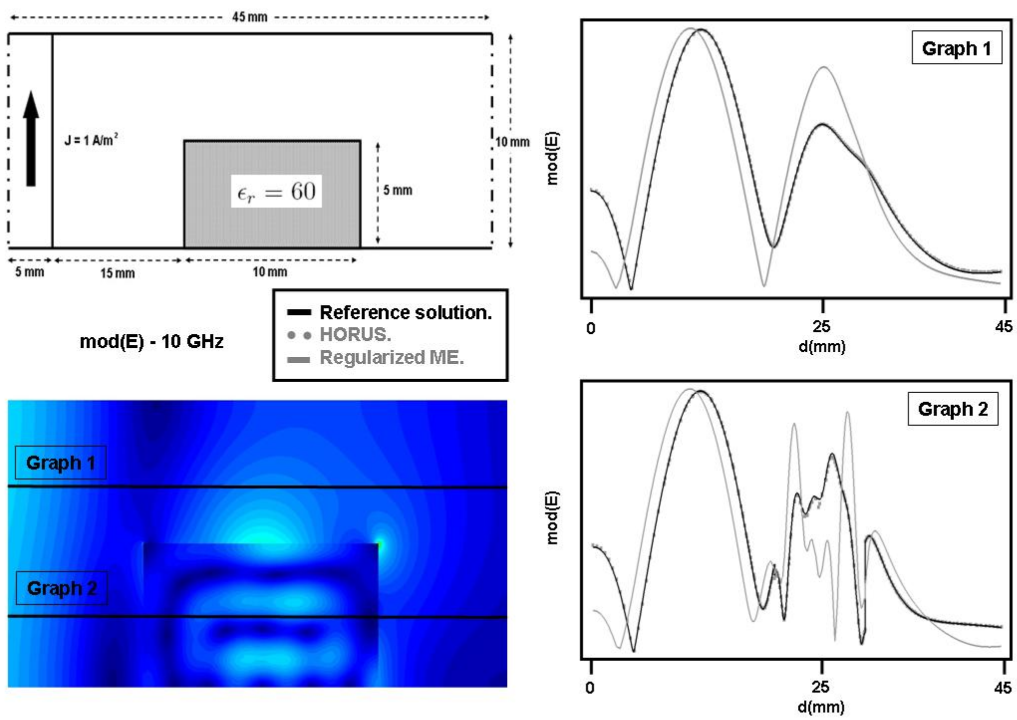

Figure 5: Dielectric in a parallel-plate waveguide.

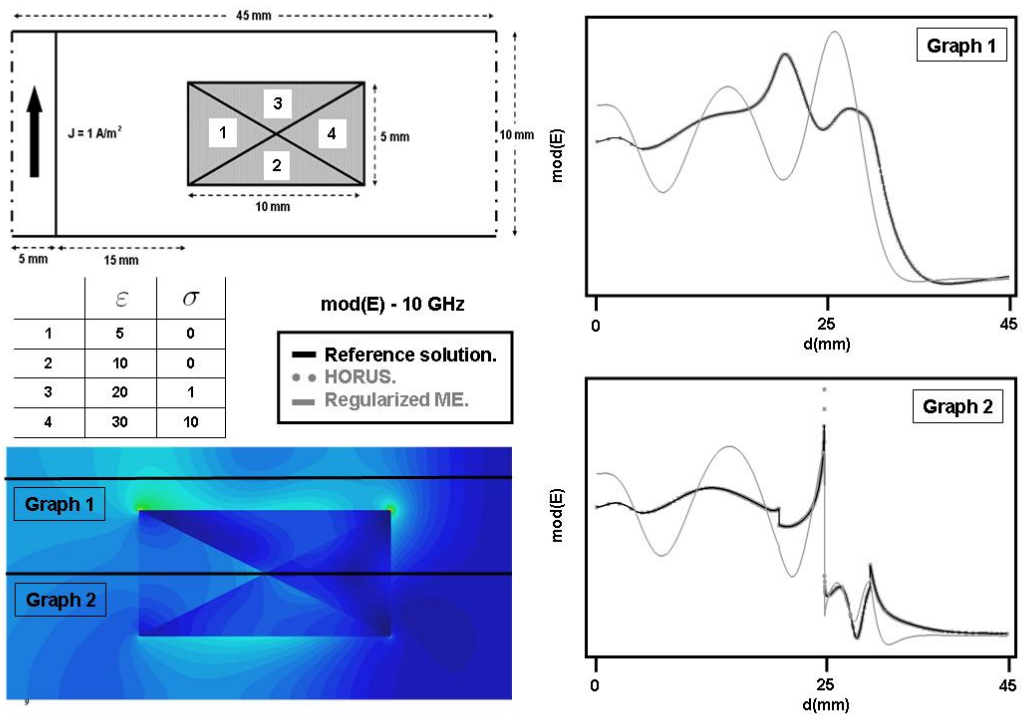

Figure 6: Intersection of dielectrics in parallel-plate waveguide.

Validation Examples in 3D

This section presents some 3D configurations solved with HORUS. Figures 7, 8 and 9 display three microwave filters taken from [23], [24] and [25]. HORUS calculates the scattering parameters of this microwave filters applying (3), (4) and (9) in the waveguide ports. The elements used were 2nd and 3rd order tetrahedral nodal elements. With 3rd order elements and 1 UL excellent results were obtained, as can be seen in the figures 7, 8 and 9. With 2nd order elements and 3 UL good results can also be obtained, but less accurate than with 3rd order elements. Nonetheless, although 2nd order elements are less accurate than 3rd order elements, they are computationally more efficient and very useful for some applications.

The last validation example (figures 10, 11, 12 and 13) is a PROCOM MU9-XP4 antenna installed in a motorbike. This problem was solved in the frame of the project PROFIT SANTTRA (*Sistema de ANTenas para Transceptores de RAdio*, Ref: FIT-330210-2006-44) based on the European project PIDEA-EUREKA SMART (*SMart Antennas system for Radio Transceivers*). EADS SN [26] performed the simulation with the Method of Moments (MoM) implemented in the commercial code FEKO [27] and the results were compared with HORUS. The objective was to calculate the Specific Absorption Rate (SAR) distribution produced by the antenna in a body placed 0.6 m away. The antenna is fed by a coaxial source with 10 W (40 dBm) at a frequency of 406 MHz. The body has a density $\rho = 1000\ Kg/m^3$ with electric properties: $\varepsilon_r = 41.3$, $\sigma = 0.58$ S/m and $\mu_r = 1$. A perfect electric conductor (PEC) ground is placed 1.22 m below the feeding point of the antenna. A PEC topcase is placed 0.15 m below the same feeding point (figures 10 and 11). HORUS employed tetrahedral nodal elements of 2nd order with 3 UL in the edges of the topcase and also 3 UL in the edges of the base and tip of the antenna. Extended 1st order ABC (8) were used as boundary conditions. The results of the simulations are shown in figures 12 and 13. The maximum SAR value obtained with HORUS was 0.06 W/Kg, located in the neck. The maximum SAR value obtained with FEKO was 0.07 W/Kg, also located in the neck. The SAR averaged over the 10 grams cube was 0.04 W/Kg (HORUS) and 0.06 W/Kg (FEKO).

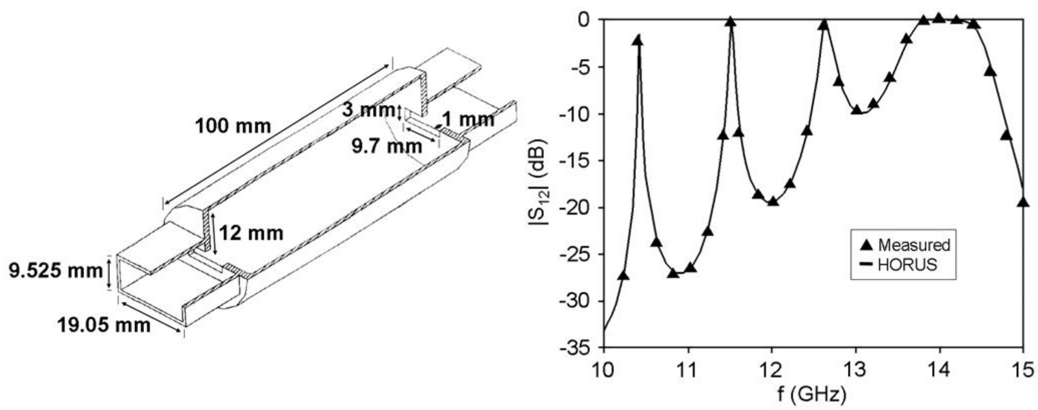

Figure 7: Cylindrical cavity filter measured in [23]. HORUS used tetrahedral 3rd order nodal elements and the weight in (14) was set to zero only in the elements with a node in contact with the edges of the coupling slots (1 UL).

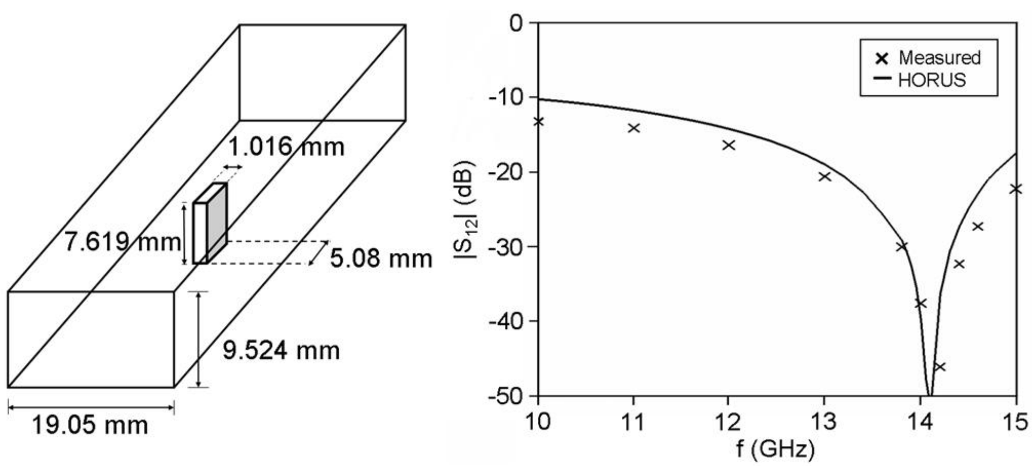

Figure 8: Ridge waveguide measured in [24]. HORUS used tetrahedral 3rd order nodal elements and the weight in (14) was set to zero only in the elements with a node in contact with the edges of the ridge (1 UL).

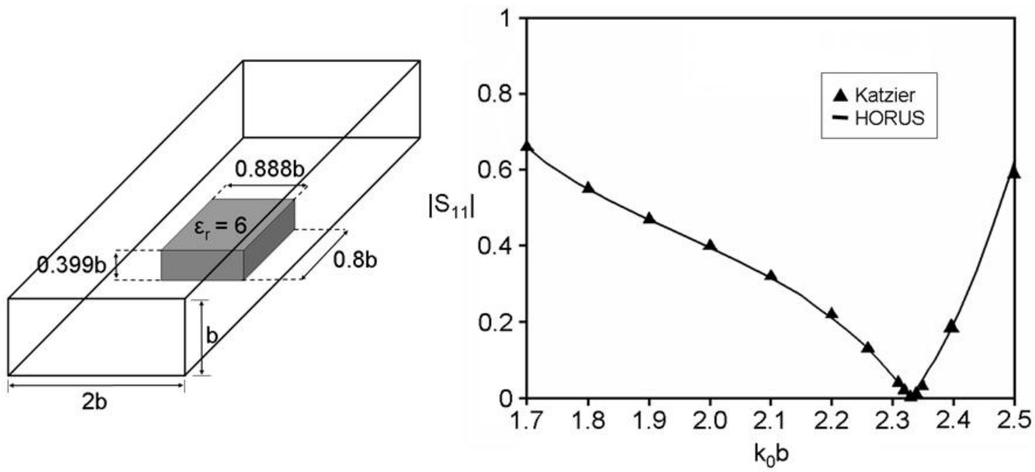

Figure 9: Dielectric in a rectangular waveguide. Reference scattering parameters values obtained from [25]. HORUS used tetrahedral 3rd order nodal elements and the weight in (14) was set to zero only in the elements with a node in contact with the edges of the dielectric (1 UL).

Figure 10: PROCOM MU9-XP4 antenna installed in a motorbike. Left: real geometry. Right: FEM mesh by GiD [28] used for HORUS simulations. The mesh is composed of tetrahedral 2nd order nodal elements.

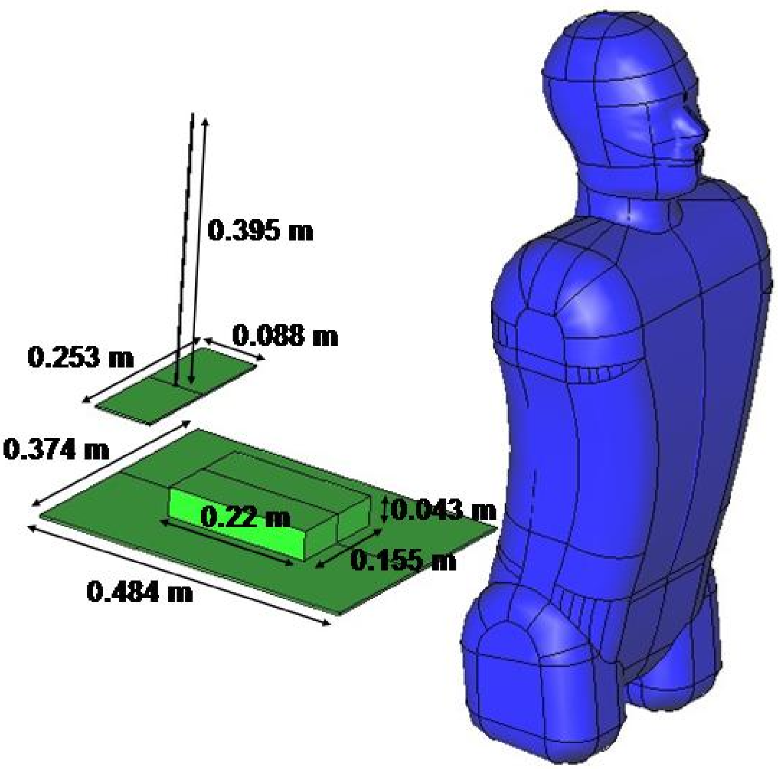

Figure 11: PROCOM MU9-XP4 antenna installed in a motorbike. Detailed geometry.

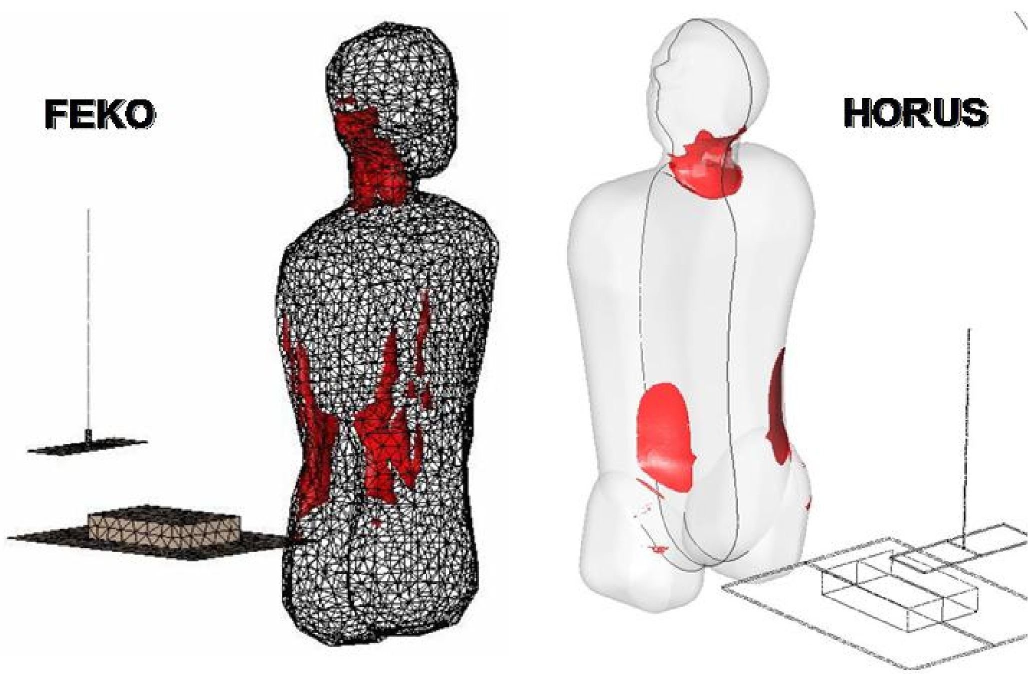

Figure 12: Iso-surfaces for SAR = 0.03 W/Kg. Left: FEKO results. Right: HORUS results.

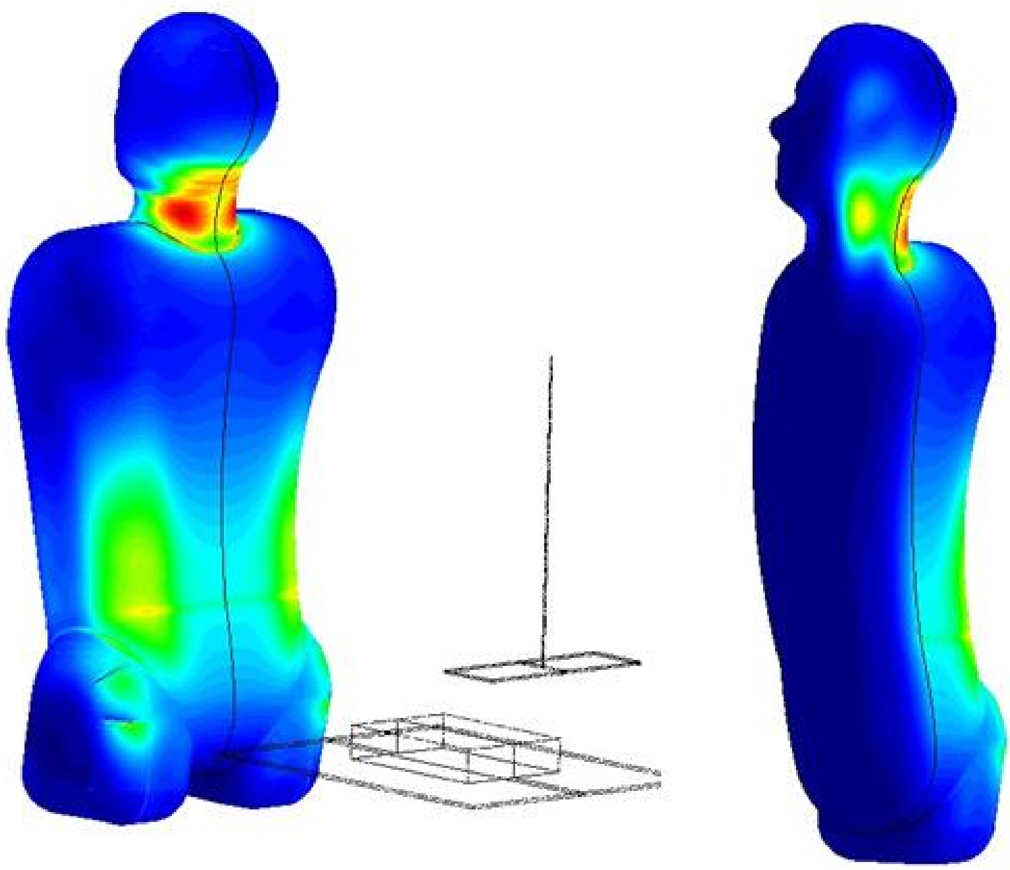

Figure 13: SAR distribution calculated with HORUS. Maximum value is 0.06 W/Kg, located in the neck.

Conclusion

It is demonstrated in this paper that accurate solution can be obtained with nodal elements and the regularized formulation even in the presence of electromagnetic field singularities. To do so, we have used a simplified version of the weighted regularized Maxwell equation method. This simplification consists of a weight equal to zero applied in the elements near a singularity and equal to one in the rest. The number of layers of elements whose weight is equal to zero depends on their size and order. For a given element order this number is fixed but, if for some reason, we mesh with very small elements near a singularity, this number must be increased to obtain a correct solution. It is necessary to carry out a further study on the relationship between the number of elements with a weight equal to zero and the element size, element order and singularity order. Also, a deeper theoretical knowledge of how this simplification affects the well-conditioning and convergence of the regularized formulation is needed. In a future work a comparative performance of HORUS with other methods will be given.

Acknowledgments

This work has been partially founded by the Spanish Ministry of Industry, Tourism and Commerce in the frame of the project PROFIT SANTTRA (*Sistema de ANTenas para Transceptores de RAdio*, Ref: FIT-330210-2006-44) based on the European project PIDEA-EUREKA SMART (*SMart Antennas system for Radio Transceivers*). The author also wants to acknowledge the help of Michael Mattes, Juan R. Mosig and Sergio López Peña from LEMA-EPFL (Lausanne, Switzerland) for their help in the validation of the code used in this article.

References

[1] J. C. Nedelec. Mixed finite elements in R^3. *Numerische Mathematik*, 35:315–341, 1980.

[2] Magdalena Salazar-Palma, Tapan K. Sarkar, Luis-Emilio García Castillo, Tanmoy Roy, and Antonije Djordjevic. *Iterative and Self-Adaptive Finite-Elements in Electromagnetic Modeling*. Artech House Publishers, 1998.

[3] J. Jin. *The Finite Element Method in Electromagnetics*. John Wiley & Sons, 2nd edition, 2002.

[4] J. P. Webb. Edge elements and what they can do for you. *IEEE Transactions on Magnetics*, 29:1460–1465, 1993.

[5] Gerrit Mur. The fallacy of edge elements. *IEEE Transactions on Magnetics*, 34:3244–3247, 1998.

[6] Gerrit Mur. Edge elements, their advantages and their disadvantages. *IEEE Transactions on Magnetics*, 30:3552–3557, 1994.

[7] I. E. Lager. *Finite Element Modelling of Static and Stationary Electric and Magnetic Fields*. PhD thesis, Delft Univesity, 1996.

[8] Patrick Ciarlet Jr. Augmented formulations for solving Maxwell equations. *Computer Methods in Applied Mechanics and Engineering*, 194:559–586, 2005.

[9] R. Hiptmair. Finite elements in computational electromagnetism. *Acta Numerica*, 1:237–339, 2002.

[10] C. Hazard and M. Lenoir. On the solution of the time-harmonic scattering problems for Maxwell's equations. *SIAM Journal on Mathematical Analysis*, 27:1597–1630, 1996.

[11] M. Costabel and M. Dauge. Weighted regularization of Maxwell equations in polyhedral domains. *Numerische Mathematik*, 93(2):239–277, 2002.

[12] K. D. Paulsen, D. R. Lynch, and J. W. Strohbehn. Three-dimensional finite, boundary, and hybrid element solutions of the Maxwell equations for lossy dielectric media. *IEEE Transactions on Microwave Theory and Techniques*, 36:682–693, 1988.

[13] Willian E. Boyse, Daniel R. Lynch, Keith D. Paulsen, and Gerald N. Minerbo. Nodal-based finite-element modeling of Maxwell's equations. *IEEE Transactions on Antennas and Propagation*, 40:642–651, 1992.

[14] Keith D. Paulsen, Daniel R. Lynch, and John W. Strohbehn. Numerical treatment of boundary conditions at points connecting more than two electrically dinstinct regions. *Communications in Applied Numerical Methods*, 3:53–62, 1987.

[15] M. Costabel. A coercive bilinear form for Maxwell's equations. *Journal of Mathematical Analysis and Applications*, 157(2):527–541, 1991.

[16] M. Costabel and M. Dauge. Maxwell and Lamé eingevalues on polyhedra. *Mathematical Methods in Applied Science*, 22:243–258, 1999.

[17] A.S. Bonnet-Ben Dhia, C. Hazard, and S. Lohrengel. A singular field method for the solution of Maxwell's equations in polyhedral domains. *SIAM Journal on Applied Mathematics*, 59(6):2028–2044, 1999.

[18] F. Assous, P. Ciarlet, S. Labrunie, and S. Lohrengel. The singular complement method. *Prepublication de l'Institut Elie Cartan*, 2001.

[19] Willian E. Boyse and Andrew A. Seidl. A hybrid finite element method for 3-D scattering using nodal and edge elements. *IEEE Transactions on Antennas and Propagation*, 42:1436–1442, 1994.

[20] Kurt Preis, Oszkár Bíró, and Igor Ticar. Gauged current vector potential and reetrant corners in the FEM analysis of 3D eddy currents. *IEEE Transactions on Magnetics*, 36:840–843, 2000.

[21] M. Kaltenbacher and S. Reitzinger. Appropiate finite-element formulation for 3-D electromagnetic-field problems. *IEEE Transactions on Magnetics*, 38:513–516, 2002.

[22] J. Van Bladel. *Singular Electromagnetic Fields and Sources*. IEEE Press, 1991.

[23] José R. Montejo-Garai and Juan Zapata. Full-wave design and realization of multicoupled dual-mode circular waveguide filters. *IEEE Transactions on Microwave Theory and Techniques*, 43(6):1290–1297, 1995.

[24] Raafat R. Mansour, Robert S. K. Tong, and Robert H. Machpie. Simplified description of the field distribution in finlines and ridge waveguides and its application to the analysis of E-plane discontinuities. *IEEE Transactions on Microwave Theory and Techniques*, 36(12):1825–1832, 1988.

[25] H. Katzier. Streuverhalten elektromagnetischer Wellen bei sprunghaften Übergängen geschirmter dielektrischer leitungen. *Arch. Elek. Übertragung*, 38:290–296, 1984.

[26] EADS SN. *http://www.eads.net*. EADS Secure Networks. Elancourt, France.

[27] FEKO. *http://www.feko.info*. EM Software & Systems.

[28] GiD:Personal Pre and Post Processor. *http://www.gidhome.com*. International Center for Numerical Methods in Engineering (CIMNE).